

## Chapter Review



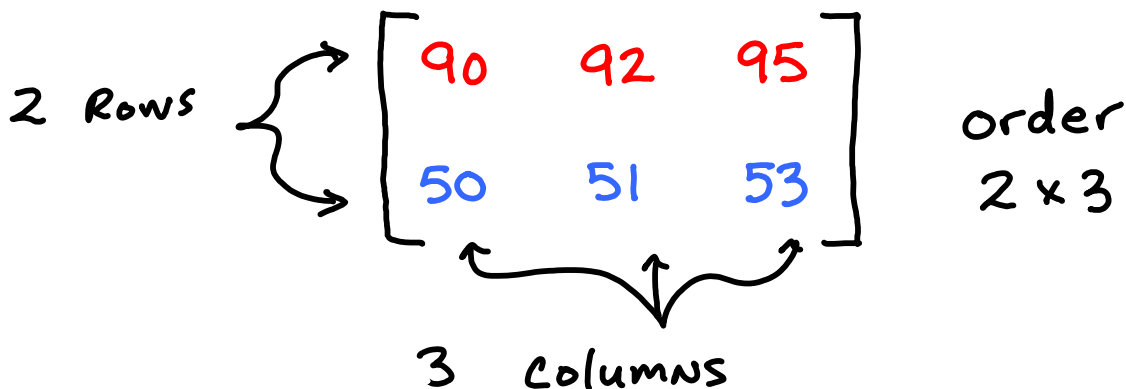
### Introduction to Matrices

a matrix is a way to organize information (data) in rows and columns

Temperature for these days

	M	W	F
Hi	90	92	95
Low	50	51	53

order is the size of a matrix





## Matrix Operations

You can  $+$ ,  $-$ ,  $\times$ ,  $\div$  matrices

however, certain conditions apply

Adding / Subtracting Matrices - must have same size.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 9 & -1 \\ 1 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 11 & 2 \\ 5 & 2 & 14 \end{bmatrix}$$

add respective entries

Scalar multiplication (a number  $\times$  matrix)

$$3 \begin{bmatrix} 1 & 5 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 15 \\ 18 & 9 \end{bmatrix}$$

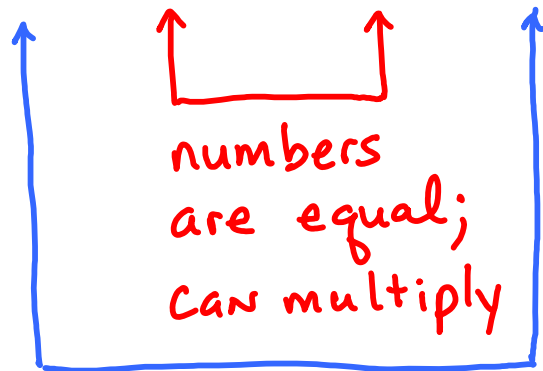
multiply 3 to all entries



# Matrix Multiplication

First, determine if two matrices can be multiplied

$$(2 \times 3) \times (3 \times 2)$$



product will be a  $2 \times 2$  matrix

## Procedure

$$\begin{array}{l} \text{use rows} \longrightarrow \\ \longrightarrow \end{array} \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} R1C1 & R1C2 \\ R2C1 & R2C2 \end{bmatrix}$$

$\uparrow \qquad \uparrow$   
use columns

$$\begin{bmatrix} 1(5) + 3(2) & 1(1) + 3(4) \\ 0(5) + -2(2) & 0(1) + -2(4) \end{bmatrix} = \begin{bmatrix} 11 & 13 \\ -4 & -8 \end{bmatrix}$$



## Determinants

A determinant is a number associated with every square matrix - it has many valuable applications

Determinant of a  $2 \times 2$  matrix

Given  $\begin{bmatrix} 5 & 3 \\ 1 & 2 \end{bmatrix}$  find  $\begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix}$

↑  
matrix

↘  
determinant

$$\begin{aligned} \text{find } \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} &= (5 \cdot 2) - (1 \cdot 3) \\ &= 10 - 3 \\ &= 7 \end{aligned}$$

Determinant of  $3 \times 3$  matrix - two methods

- Expansion Method
- Diagonals Method

# Expansion Method

use 2x2 determinants

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & 2 \\ 1 & 0 & 3 \end{bmatrix}$$

find

$$\begin{vmatrix} 1 & 2 & 4 \\ 3 & 5 & 2 \\ 1 & 0 & 3 \end{vmatrix}$$

Example - expand along row 1 - see pattern

$$\begin{vmatrix} 1 & 2 & 4 \\ 3 & 5 & 2 \\ 1 & 0 & 3 \end{vmatrix} = 1 \begin{vmatrix} 5 & 2 \\ 0 & 3 \end{vmatrix} = 1(15 - 0) = 15$$

$$- \begin{vmatrix} 1 & 2 & 4 \\ 3 & 5 & 2 \\ 1 & 0 & 3 \end{vmatrix} = -2 \begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} = -2(9 - 2) = -14$$

$$+ \begin{vmatrix} 1 & 2 & 4 \\ 3 & 5 & 2 \\ 1 & 0 & 3 \end{vmatrix} = 4 \begin{vmatrix} 3 & 5 \\ 1 & 0 \end{vmatrix} = 4(0 - 5) = -20$$

$$\text{Determinant} = 15 + (-14) + (-20) = -19$$

## 🚩 Sign pattern along rows

$$\begin{array}{ccc|ccc} 1 & 2 & 4 & + & - & + \\ 3 & 5 & 2 & - & + & - \\ 1 & 0 & 3 & + & - & + \end{array}$$

try to expand  
along rows  
with the most  
zeros

## Diagonal Method

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & 2 \\ 1 & 0 & 3 \end{bmatrix} \quad \text{find} \quad \begin{vmatrix} 1 & 2 & 4 \\ 3 & 5 & 2 \\ 1 & 0 & 3 \end{vmatrix}$$

step 1 - rewrite the first 2 columns after

$$\begin{array}{ccc|cc} 1 & 2 & 4 & 1 & 2 \\ 3 & 5 & 2 & 3 & 5 \\ 1 & 0 & 3 & 1 & 0 \end{array} =$$

determinant

first 2 columns

Second, subtract these products

$$\begin{array}{ccc|cc} 1 & 2 & 4 & 1 & 2 \\ 3 & 5 & 2 & 3 & 5 \\ 1 & 0 & 3 & 1 & 0 \end{array} =$$

First, add these products

$$\begin{aligned} & \underline{(15 + 4 + 0)} - \underline{(20 + 0 + 18)} = \\ & \underline{19} - \underline{38} = -19 \end{aligned}$$



## Identity and Inverse Matrices

$$a \cdot 1 = a \quad \leftarrow \begin{array}{l} 1 \text{ is the} \\ \text{identity} \end{array}$$

$$7 \cdot 1 = 7$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$$

↑  
identity matrix (2 × 2)

## Inverse of a 2 x 2 matrix

Given

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Inverse

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - cb} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\text{note} = ad - cb \neq 0$$

Example

$$A = \begin{bmatrix} 4 & 1 \\ 2 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\begin{vmatrix} 4 & 1 \\ 2 & -2 \end{vmatrix}} \begin{bmatrix} \textcircled{4} & \textcircled{1} \\ \textcircled{2} & \textcircled{-2} \end{bmatrix}$$

opposite

FLIP

Determinant

$$A^{-1} = \frac{1}{-8 - 2} \begin{bmatrix} -2 & -1 \\ -2 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-10} \begin{bmatrix} -2 & -1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{2}{10} & \frac{1}{10} \\ \frac{2}{10} & -\frac{4}{10} \end{bmatrix}$$



## Solving Systems using Inverse Matrices

Formula

$$A \cdot X = B$$

$$X = A^{-1}B$$

$$\begin{cases} 5x + 4y = 6 \\ -2x - 3y = -1 \end{cases}$$

$$\begin{bmatrix} 5 & 4 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

→ Find inverse

$$A = \begin{bmatrix} 5 & 4 \\ -2 & -3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-15+8} \begin{bmatrix} -3 & -4 \\ 2 & 5 \end{bmatrix}$$

$$A^{-1} = -\frac{1}{7} \begin{bmatrix} -3 & -4 \\ 2 & 5 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3/7 & 4/7 \\ -2/7 & -5/7 \end{bmatrix}$$

$$X = A^{-1} \cdot B$$

$$\begin{bmatrix} 3/7 & 4/7 \\ -2/7 & -5/7 \end{bmatrix} \begin{bmatrix} 6 \\ -1 \end{bmatrix} =$$

(2 × 2)    (2 × 1)

$$X = A^{-1} \cdot B$$
$$\begin{bmatrix} 3/7 & 4/7 \\ -2/7 & -5/7 \end{bmatrix} \begin{bmatrix} 6 \\ -1 \end{bmatrix} = \begin{bmatrix} (18/7 + -4/7) \\ (-12/7 + 5/7) \end{bmatrix}$$

(2 × 2)      (2 × 1)      (2 × 1)

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 14/7 \\ -7/7 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad (2, -1)$$

solution





## Solving Systems using Cramer's Rule

Another way to solve systems using determinants - the formulas look complex but the rule is easy to use - see example

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases} \quad \begin{matrix} x = \frac{\begin{vmatrix} c & b \\ f & e \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}} \\ y = \frac{\begin{vmatrix} a & c \\ d & f \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}} \end{matrix}$$

2 x 2 system

$$3 \times 3 \text{ system} \quad \begin{cases} ax + by + cz = d \\ ex + fy + gz = h \\ ix + jy + kz = l \end{cases}$$

$$x = \frac{\begin{vmatrix} d & b & c \\ h & f & g \\ l & j & k \end{vmatrix}}{\begin{vmatrix} a & b & c \\ e & f & g \\ i & j & k \end{vmatrix}} \quad y = \frac{\begin{vmatrix} a & d & c \\ e & h & g \\ i & l & k \end{vmatrix}}{\begin{vmatrix} a & b & c \\ e & f & g \\ i & j & k \end{vmatrix}} \quad z = \frac{\begin{vmatrix} a & b & d \\ e & f & h \\ i & j & l \end{vmatrix}}{\begin{vmatrix} a & b & c \\ e & f & g \\ i & j & k \end{vmatrix}}$$



## Solving systems using **Cramer's Rule**

$$\begin{cases} -2x + y = 8 \\ 3x + y = -2 \end{cases} \quad (-2, 4)$$

notice  
the x  
column  
was replaced

$$x = \frac{\begin{vmatrix} 8 & 1 \\ -2 & 1 \end{vmatrix}}{\begin{vmatrix} -2 & 1 \\ 3 & 1 \end{vmatrix}}$$

$$y = \frac{\begin{vmatrix} -2 & 8 \\ 3 & -2 \end{vmatrix}}{\begin{vmatrix} -2 & 1 \\ 3 & 1 \end{vmatrix}}$$

notice  
the y  
column  
was  
replaced

the coefficient of the x and y columns

Calculate the determinants and simplify

$$x = \frac{10}{-5} \quad y = \frac{-20}{-5}$$

$$x = -2 \quad y = 4$$

$$(-2, 4)$$

$$\begin{cases} 3x + 2y + 4z = 11 \\ 2x - y + 3z = 4 \\ 5x - 3y + 5z = -1 \end{cases} \quad \begin{vmatrix} 3 & 2 & 4 \\ 2 & -1 & 3 \\ 5 & -3 & 5 \end{vmatrix} = 18$$

$$x = \frac{\begin{vmatrix} 11 & 2 & 4 \\ 4 & -1 & 3 \\ -1 & -3 & 5 \end{vmatrix}}{18} = \frac{-54}{18} = -3$$

$$y = \frac{\begin{vmatrix} 3 & 11 & 4 \\ 2 & 4 & 3 \\ 5 & -1 & 5 \end{vmatrix}}{18} = \frac{36}{18} = 2$$

$$z = \frac{\begin{vmatrix} 3 & 2 & 11 \\ 2 & -1 & 4 \\ 5 & -3 & -1 \end{vmatrix}}{18} = \frac{72}{18} = 4$$

solution  $(-3, 2, 4)$